In the class, I reverse some trivial inequality. (Sorry)

Correction on tutorial class 18/4

Example 0.1. *Given* $f \in R[-2,2]$. *Define* $F : (-1,1) \to \mathbb{R}$ *by*

$$F(h) = \int_0^1 h|f(x+h) - f(x)|\,dx.$$

Then F *is differentiable at* $h = 0$.

Proof. Without loss of generality, we consider $h \to 0^+$. The idea below follows from the Lebesgue's Criterion for Riemann integrability.

Let $\epsilon > 0$, there exists $\mathcal{P} : -2 = x_0 < x_1 < ... < x_N = 2$ partition on $[-2,2]$ such that

$$\sum_{i=1}^{N}(M_i - m_i)\Delta x_i < \epsilon^2. \tag{1}$$

We splits the set of intervals $[x_j, x_{j+1}]$ into two parts. Denote A to be those intervals in $[0,1]$ such that

$$M_j - m_j < \epsilon.$$

And B to be those intervals in $[0,1]$ such that

$$M_j - m_j \geq \epsilon.$$

By (1), we have

$$\sum_B \Delta x_j < \epsilon^{-1}\sum_B (M_j - m_j)\Delta x_j < \epsilon.$$

Since $f \in R[-2,2]$, we may assume f is bounded above by $M > 0$. For such partition $\mathcal{P}$,

$$\int_B |f(x+h) - f(x)|\,dx \leq 2M\epsilon.$$

On A, we have $|f(x) - f(y)| < \epsilon$ for any $x, y \in [x_j, x_{j+1}]$, where $[x_j, x_{j+1}] \in A$. Choose $\delta = \epsilon N^{-1} \min\{\Delta x_j : j = 1, 2, ..., N\} > 0$. Then for $[x_j, x_{j+1}] \in A$,

$$\begin{aligned}\int_{x_j}^{x_{j+1}} |f(x+h) - f(x)|\,dx &= \int_{x_j}^{x_{j+1}-h} |f(x+h) - f(x)|\,dx + \int_{x_{j+1}-h}^{x_{j+1}} |f(x+h) - f(x)|\,dx \\ &< \epsilon \cdot \Delta x_j + 2M \cdot h.\end{aligned}$$

Therefore, if $h \in (0, \delta)$

$$\int_A |f(x+h) - f(x)|\,dx < \epsilon + 2M \cdot hN \leq \epsilon(2M+1).$$

To conclude, we show that for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $h \in (0, \delta)$,

$$\int_0^1 |f(x+h) - f(x)|\,dx < \epsilon(4M+1).$$

□

The above approach is making use of the fact that Riemann integrable function is almost continuous function in the sense of integration. Another way to observe Riemann integrable function is via step function as shown in the followings.

Proof. We first assume f is a step function defined on $[-2,2]$. We write

$$f = \sum_{i=1}^{n} \alpha_i \chi_{I_i} \tag{2}$$

where $I_i = [x_i, x_{i+1})$ for $i = 1, ..., n-1$. And $I_n = [x_{n-1}, x_n]$. We can easily observed that it suffices to consider $f = \chi_I$ where I is some interval (closed or half closed). Let $I = [a,b] \subset [-2,2]$. Then

$$\int_0^1 |f(x+h) - f(x)|\, dx \leq \int_{a-h}^{a} dx + \int_b^{b+h} dx = 2h \to 0 \text{ as } h \to 0.$$

If f is given by (2). We denote $f_i = \chi_{I_i}$, then

$$\begin{aligned}\int_0^1 |f(x+h) - f(x)|\, dx &= \int_0^1 \left| \sum_{i=1}^{n} \alpha_i \big[f_i(x+h) - f_i(x)\big] \right| dx \\ &\leq \sum_{i=1}^{n} |\alpha_i| \left(\int_0^1 |f_i(x+h) - f_i(x)|\, dx \right).\end{aligned}$$

By the above argument, we see that

$$\int_0^1 |f(x+h) - f(x)|\, dx \to 0 \text{ as } h \to 0.$$

So the case of step function have been verified. Now we aim to approximate integrable function by step functions. As f is integrable, there exists a partition such that the upper integral $0 \leq U(f, \mathcal{P}) - \int_{-2}^{2} f < \epsilon$. That is to say that we can find a step function $g \geq f$ on $[-2,2]$ such that

$$0 \leq \int_{-2}^{2} (g - f)\, dx < \epsilon.$$

As g is step function, there exists $\delta > 0$ such that if $|h| < \delta$,

$$\int_0^1 |g(x+h) - g(x)| < \epsilon.$$

Combined all these estimates, we conclude that if $|h| < \delta$,

$$\begin{aligned}\int_0^1 |f(x+h) - f(x)|\, dx &\leq \int_0^1 |g(x+h) - g(x)|\, dx + \int_0^1 |f(x+h) - g(x+h)|\, dx \\ &\quad + \int_0^1 |f(x) - g(x)|\, dx \\ &< \epsilon + \int_0^1 |f(x) - g(x)|\, dx + \int_h^{1+h} |f(x) - g(x)|\, dx \\ &< \epsilon + 2\epsilon = 3\epsilon.\end{aligned}$$

□